TFb Soundness

Weakening : $\Gamma \vdash e : \tau$ and x not free in e then $\Gamma, x : \tau' \vdash e : \tau$.

Substitution : $\Gamma, x : \tau_1 \vdash e : \tau_2$ and $\Gamma \vdash v : \tau_1$ then $\Gamma \vdash e[v/x] : \tau_2$.

Soundness in TFb

$v \in \tau$

If $\underbrace{\Gamma \vdash e : \tau}_{\text{given}}$ and $\underbrace{e \Rightarrow v}_{\text{given}}$ then $\underbrace{\Gamma \vdash v : \tau}_{\text{produce}}$.

Proof. By induction on the height of $e \Rightarrow v$ and then by case analysis on the proof rule used.

Base Case. Proof tree has height 1 and so is an axiom.
So $e \Rightarrow v$ uses the Value Rule.
So $e = v$. So since $\Gamma \vdash e : \tau$,
$\Gamma \vdash v : \tau$.

Therefore,
v is one of $\mathbb{Z}$, $\mathbb{B}$, or a function.
- If $v \in \mathbb{Z}$ then $\Gamma \vdash e : \tau$ must use the Int Rule. Therefore, $\tau = \text{Int}$. We must then show that $\Gamma \vdash \mathbb{Z} : \text{Int}$, which is true by the Int Rule.
- If $v \in \mathbb{B}$, this proceeds as in the Int case.
- If v is a function, this proceeds as in the Int case.

Inductive Step. Proof tree of $e \Rightarrow v$ has height > 1.

$$\frac{\Gamma \vdash e : \text{Bool}}{\Gamma \vdash \text{Not } e : \text{Bool}}$$

- If the Not Rule is used, then $v \in \mathbb{B}$, $e = \text{Not } e'$. Then the proof $\Gamma \vdash e : \tau$ must have used the Not Rule so $\tau = \text{Bool}$. So by the Boolean Rule, $\Gamma \vdash v : \tau$.
- If the Plus Rule is used, then $v \in \mathbb{Z}$, $e = e_1 + e_2$. Then the proof of $\Gamma \vdash e : \tau$ must have used the Plus Rule, so $\tau = \text{Int}$. So by the Int Rule, $\Gamma \vdash v : \tau$.
- All binary operators proceed as above.
- If the If True Rule is used, then $e = \text{If } e_1 \text{ Then } e_2 \text{ Else } e_3$. Also $e_1 \Rightarrow \text{True}$ and $e_2 \Rightarrow v$. Since $e = \text{If } e_1 \text{ Then } e_2 \text{ Else } e_3$, the proof of $\Gamma \vdash e : \tau$ must use the If Rule. So $\Gamma \vdash e_1 : \text{Bool}$ and $\Gamma \vdash e_2 : \tau$. Since $e_2 \Rightarrow v$ has lesser height than $e \Rightarrow v$, by ind hyp. $\Gamma \vdash v : \tau$.
- If the If False Rule is used, then the same strategy applies.

- In the case that $e \Rightarrow v$ uses the Application Rule, $e_1 \Rightarrow \text{Function } x \rightarrow e'$ and $e_2 \Rightarrow v_2$ and $e'[v_2/x] \Rightarrow v$. Also $e = e_1\ e_2$. Therefore the Application Rule was used in $\Gamma \vdash e : \tau$. So $\Gamma \vdash e_1 : \tau' \rightarrow \tau$ and $\Gamma \vdash e_2 : \tau'$. Since there is a proof of $\Gamma \vdash e_1 : \tau' \rightarrow \tau$ and e_1 is a function, that proof must have used the Function Rule. Then $\Gamma, x : \tau' \vdash e' : \tau$. Because $e_2 \Rightarrow v_2$, $\Gamma \vdash e_2 : \tau'$, and the height of $e_2 \Rightarrow v_2$ is less than the height of $e \Rightarrow v$, by ind. hyp. $\Gamma \vdash v_2 : \tau'$. Because $\Gamma, x : \tau' \vdash e' : \tau$ and $\Gamma \vdash v_2 : \tau'$, then by Substitution Lemma, $\Gamma \vdash e'[v_2/x] : \tau$. Because $e'[v_2/x] \Rightarrow v$ has lesser height than $e \Rightarrow v$ and $\Gamma \vdash e'[v_2/x] : \tau$, by ind hyp. $\Gamma \vdash v : \tau$. This case is finished.
- Let case proceeds similarly (but is easier).

QED

To show that stuck expressions don't typecheck: Fb$\overset{!!}{\frown}$

1. Extend value space: $v ::= \cdots \mid \overset{!!}{\frown}$
2. Extend operational semantics to formalize stuck cases.

$$\frac{e_1 \Rightarrow \not\mathbb{Z} \qquad e_2 \Rightarrow \mathbb{B}}{e_1 + e_2 \Rightarrow \overset{!!}{\frown}}$$

3. Leave the type system alone.

Claim: If $\Gamma \vdash e : \tau$ then $e \not\Rightarrow \overset{!!}{\frown}$.

Proof: By soundness and because $\Gamma \vdash \overset{!!}{\frown} : \tau'$ does not hold for any τ'.