TFb Soundness

If $e \Rightarrow v$ and $\emptyset \vdash e : \tau$ then $v \in \tau$.

We're doing this next time.

TFb Weaking Lemma

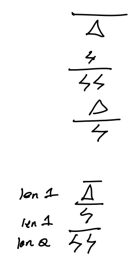
$\forall e, \tau, \Gamma, x.$ If $\Gamma \vdash e : \tau$ and x not free in e then $\Gamma, x : \tau' \vdash e : \tau$.

given (input) — given (input) — want to show (output)

By induction on the height of the proof of $\Gamma \vdash e : \tau$ and then by case analysis on the proof rule.

Base Case: When height of proof tree is 1, the proof must use an axiom. The axioms of this proof system are: Integer, Bool, and Var.

- If the Integer Rule is used, then $e \in \mathbb{Z}$ and $\tau = \text{Int}$. Then by Integer Rule, $\Gamma, x : \tau' \vdash e : \tau$.
- If the Bool Rule is used, then $e \in \mathbb{B}$ and $\tau = \text{Bool}$. Then by Bool Rule, $\Gamma, x : \tau' \vdash e : \tau$.
- If the Var Rule is used, then $e = x'$. Because x not free in e, $x' \neq x$. So by Var Rule, $\Gamma, x : \tau' \vdash e : \tau$.



Inductive Step: The height of proof tree is greater than 1.

In "Let a=5 In a+1" a is not free.

In "a+1" a is free.

Let a=5 In 4 — a is not free

Let a=5 In a — a is free — a is not free

Let x'=5 In 4

Let x'=5 In x

- If the If Rule is used, then the $e = \text{If } e_1 \text{ Then } e_2 \text{ Else } e_3$. Also, $\Gamma \vdash e_1 : \text{Bool}$ and $\Gamma \vdash e_2 : \tau$ and $\Gamma \vdash e_3 : \tau$. Because x not free in e, x not free in e_1, e_2, or e_3. By ind hyp., $\Gamma, x : \tau' \vdash e_1 : \text{Bool}$, $\Gamma, x : \tau' \vdash e_2 : \tau$, and $\Gamma, x : \tau' \vdash e_3 : \tau$. By If Rule, $\Gamma, x : \tau' \vdash \text{If } e_1 \text{ Then } e_2 \text{ Else } e_3 : \tau$.
- If the Let Rule is used, then $e =$ "$\text{Let } x' : \tau'' = e_1 \text{ In } e_2$". Because x is not free in e, x is not free in e_1. There are two cases: either $x = x'$ or x is not free in e_2.
  - In the case $x \neq x'$, then x is not free in e_2. Because the Let Rule is used, $\Gamma \vdash e_1 : \tau''$ and $\Gamma, x' : \tau'' \vdash e_2 : \tau$. By ind hyp. $\Gamma, x : \tau' \vdash e_1 : \tau''$ and $\Gamma, x' : \tau'', x : \tau' \vdash e_2 : \tau$. By Let Rule, $\Gamma, x : \tau' \vdash (\text{Let } x' : \tau'' = e_1 \text{ In } e_2) : \tau$.
  - Otherwise, $x = x'$. Because Let Rule is used, $\Gamma \vdash e_1 : \tau''$ and $\Gamma, x : \tau'' \vdash e_2 : \tau$. By ind hypothesis, $\Gamma, x : \tau' \vdash e_1 : \tau''$. We want to show that $\Gamma, x : \tau', x : \tau'' \vdash e_2 : \tau$. By defn, $\Gamma, x : \tau'' = \Gamma, x : \tau', x : \tau''$. So $\Gamma, x : \tau', x : \tau'' \vdash e_2 : \tau$. By Let Rule, $\Gamma, x : \tau' \vdash e : \tau$.
- and so on ...

TFb Substitution Lemma

If $\Gamma, x : \tau_1 \vdash e : \tau_2$ and $\Gamma \vdash v : \tau_1$ then $\Gamma \vdash e[v/x] : \tau_2$.

Proof. By induction on height of $\Gamma, x : \tau_1 \vdash e : \tau_2$ and then by case analysis on the proof rule.

Base case: In the base case, $\Gamma, x : \tau_1 \vdash e : \tau_2$ has height 1.

- If Integer Rule is used, then $e \in \mathbb{Z}$ and $\tau_2 = \text{Int}$.
  So $e[v/x] = e$. Therefore, $\Gamma, x : \tau_1 \vdash e[v/x] : \tau_2$.
  By Strengthening Lemma, $\Gamma \vdash e[v/x] : \tau_2$ and this case is finished.
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Inductive Step: In the inductive step, proof height is greater than 1.

- If the If Rule is used then $e = \text{If } e_1 \text{ Then } e_2 \text{ Else } e_3$.
  Also, $\Gamma, x : \tau_1 \vdash e_1 : \text{Bool}$ and $\Gamma, x : \tau_1 \vdash e_2 : \tau_2$ and $\Gamma, x : \tau_1 \vdash e_3 : \tau_2$.
  By ind hyp., $\Gamma \vdash e_1[v/x] : \text{Bool}$ and $\Gamma \vdash e_2[v/x] : \tau_2$ and $\Gamma \vdash e_3[v/x] : \tau_2$.
  Observe $(\text{If } e_1 \text{ Then } e_2 \text{ Else } e_3)[v/x] = \text{If } e_1[v/x] \text{ Then } e_2[v/x] \text{ Else } e_3[v/x]$.
  So by If Rule, $\Gamma \vdash e[v/x] : \tau_2$.
- If the Function Rule is used, then $e = \text{Function } x' \rightarrow e'$. There are two cases: $x = x'$ or $x \neq x'$.
  - If $x = x'$ then $e[v/x] = e$. So $\Gamma, x : \tau_1 \vdash e[v/x] : \tau_2$. By Strengthening Lemma, $\Gamma \vdash e[v/x] : \tau_2$.
  - If $x \neq x'$ then (continue as above cases)