Last time

* Proofs by induction (on the height of a tree)
* BOOL normalizing (for every BOOL e, there is some BOOL v s.t. $e \Rightarrow v$)

$$e ::= v \mid \text{Not } e \mid e \text{ And } e \mid e \text{ Or } e$$
$$v ::= \text{True} \mid \text{False}$$

To prove normalization: by induction on height of e and then case analysis on form of e

BOOL is deterministic

$$e ::= v \mid \text{Not } e \mid e \text{ And } e \mid e \text{ Or } e$$
$$v ::= \text{True} \mid \text{False}$$

$\forall e$. If $\underbrace{e \Rightarrow v_1 \text{ and } e \Rightarrow v_2}_{\text{given}}$ then $\underbrace{v_1 = v_2}_{\text{provide}}$.

Proof.

By induction on the height of e and by case analysis on form of e.

$P(1)$ Base case: height of e is 1.

There are two e with height 1: "True" and "False". If e is "True", then the proof of $e \Rightarrow v_1$ must use the Value Rule. So $e \Rightarrow v_1$ is proven by $\dfrac{}{\text{True} \Rightarrow \text{True}}$. Likewise, $e \Rightarrow v_2$ is proven by $\dfrac{}{\text{True} \Rightarrow \text{True}}$. So $v_1 = v_2 = \text{True}$. If e is "False", then this case proceeds in the same way. The base case is therefore finished. no new information

$P(1) \ldots P(n)$ implies $P(n+1)$ Inductive step: height of e is $n+1$.

There are three forms of e which apply: Not, And, and Or.

* If e is of form Not e' then e' is of height n. By ind. hyp., any two proofs of evaluation for e' yield same value. Since $e = \text{Not } e'$, the only rule which applies to e is Not Rule. Thus, $e \Rightarrow v_1$ must be of form $\dfrac{e' \Rightarrow v'_1}{\text{Not } e' \Rightarrow v_1}$ and so $e' \Rightarrow v'_1$. Likewise, $e \Rightarrow v_2$ has proven premise $e' \Rightarrow v'_2$. . . . . Since $e' \Rightarrow v'_1$ and $e' \Rightarrow v'_2$, by inductive hypothesis, $v'_1 = v'_2$. So by Not Rule, $v_1 = v_2$.
* If e is of form e' And e'' then e' and e'' have height less than e. Since e is of this form, the proof of $e \Rightarrow v_1$ must use And Rule. By its premises, $e' \Rightarrow v'_1$ and $e'' \Rightarrow v''_1$. Likewise, $e \Rightarrow v_2$ uses And Rule. By its premises, $e' \Rightarrow v'_2$ and $e'' \Rightarrow v''_2$. By ind hyp, $v'_1 = v'_2$. By ind hyp, $v''_1 = v''_2$. Because both proofs use Not Rule, v_1 = logical and of v'_1 and v''_1 = logical and of v'_2 and v''_2 = v_2. This case is finished.
* If e is of form e' Or e'', proof proceeds as in the And case.

$$\text{And}\ \dfrac{e' \Rightarrow v'_1 \qquad e'' \Rightarrow v''_1 \qquad v_1 = \text{AND of } v'_1 \text{ and } v''_1}{e' \text{ And } e'' \Rightarrow v_1} \qquad \text{And}\ \dfrac{e' \Rightarrow v'_2 \qquad e'' \Rightarrow v''_2 \qquad v_2 = \text{AND of } v'_2 \text{ and } v''_2}{e' \text{ And } e'' \Rightarrow v_2} \qquad \begin{array}{l} v'_1 = v'_2 \\ v''_1 = v''_2 \end{array}$$

TFb Strengthening Lemma

If $\underbrace{\Gamma, x:\tau' \vdash e : \tau}_{\text{given}}$ and $\underbrace{x \text{ is not free in } e}_{\text{given}}$, then $\underbrace{\Gamma \vdash e:\tau}_{\text{show}}$.

If $\{a : \text{Int}\} \vdash 5 : \text{Int}$ then $\emptyset \vdash 5 : \text{Int}$.

$$\dfrac{\dfrac{1 \Rightarrow 1 \quad 2 \Rightarrow 2}{1+2 \Rightarrow 3} \quad 0 \Rightarrow 0}{1+2=0 \Rightarrow \text{False}}$$

Proof.

By induction on height of the proof tree of $\Gamma, x:\tau' \vdash e:\tau$ and then by case analysis on the proof rule used.

Base case: height of proof is 1. Cases are Int rule, the Bool rule, and Var rule.

* If proof uses Int Rule, then it is of form $\overline{\Gamma, x:\tau' \vdash \text{[illegible]} : \text{Int}}$. So by Int rule, $\overline{\Gamma \vdash \text{[illegible]} : \text{Int}}$.
* If proof uses Bool Rule, then we proceed as in the Int Rule.
* If proof uses Var Rule then it is of form $\overline{\Gamma, x:\tau' \vdash x' : \tau}$. But $x \neq x'$ because if $x = x'$ then x is free in e. Therefore, $(x' : \tau) \in \Gamma$. So, by Var rule, $\overline{\Gamma \vdash x' : \tau}$,

Ind case: * If proof uses Not Rule. Then $e = \text{Not } e'$ and proof gives us that $\Gamma, x:\tau' \vdash e' : \text{Bool}$ and $\tau = \text{Bool}$. By ind hyp, $\Gamma \vdash e' : \text{Bool}$. So by Not Rule, $\Gamma \vdash \text{Not } e' : \text{Bool}$ and so $\Gamma \vdash e : \text{Bool}$. So this case is finished.

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