Tuesday

- ⋆ Induction
- ⋆ BOOL is normalizing
- ⋆ Goof — ~~F~~ BT was empty or node w/ 2 children which are ~~F~~ BTs of same height

BOOL normalizing

$\forall e . \exists v . \ e \Rightarrow v$

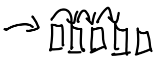
Case analysis on form e:

Induction on height of e:

$$\frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2 \qquad v = \ldots}{e_1 \text{ And } e_2 \Rightarrow v}$$

Weak vs. Strong Induction

$$\dfrac{\begin{array}{l} P(0) \\ P(n) \text{ implies } P(n+1). \end{array}}{P(n) \text{ for all } n}$$



$$\dfrac{\begin{array}{l} P(0) \\ P(0) \ldots P(n) \text{ implies } P(n+1) \end{array}}{P(n) \text{ for all } n}$$

$$\overbrace{\underbrace{(\text{True And True})}_{1} \text{ And } \underbrace{\text{True}}_{0}}^{2}$$

BOOL is deterministic

$\forall e$. If $\underbrace{e \Rightarrow v \text{ and } e \Rightarrow v'}_{\text{conditions}}$ then $\underbrace{v = v'}_{\text{trying to prove}}$.

$e ::= v \mid e \text{ And } e \mid e \text{ Or } e \mid \text{Not } e$

$v ::= \text{True} \mid \text{False}$

By induction on height of e, then by case analysis on the form of e.

$P(n) \equiv$ for e of height n, If $e \Rightarrow v$, $e \Rightarrow v'$, then $v = v'$.

P(0): height of e is 0. Therefore, e is True or e is False.

Suppose e is True. e is a value. The only rule which evaluates e is Value Rule. So, first proof is $\overline{\text{True} \Rightarrow \text{True}}$ and second proof is $\overline{\text{True} \Rightarrow \text{True}}$. Therefore, in this case, $v = v' = \text{True}$.

False proceeds the same way.

P(0).... P(n) implies P(n+1)

e has three cases: And, Or, Not.

✓ In And case, $e = e_1 \text{ And } e_2$. So $e \Rightarrow v$ must use And Rule. Then $e_1 \Rightarrow v_1$ and $e_2 \Rightarrow v_2$. Likewise $e \Rightarrow v'$ and so $e_1 \Rightarrow v_1'$ and $e_2 \Rightarrow v_2'$. e_1 has lesser height than e. By induction, $v_1 = v_1'$ and $v_2 = v_2'$. Then first proof concludes v is logical and of v_1 and v_2; second proof concludes v' is logical and of v_1 and v_2; so $v = v'$.

✓ In Or case .....

✓ In Not case ....

□

TFb

$\Gamma \vdash e : \tau$

$$\dfrac{\dfrac{}{\emptyset \vdash 1 : \mathrm{Int}} \qquad \dfrac{\dfrac{}{\{x:\mathrm{Int}\} \vdash x : \mathrm{Int}} \quad \dfrac{}{\{x:\mathrm{Int}\} \vdash 1 : \mathrm{Int}}}{\{x:\mathrm{Int}\} \vdash x+1 : \mathrm{Int}}}{\emptyset \vdash \mathrm{Let}\ x = 1\ \mathrm{In}\ x+1 : \mathrm{Int}}$$

(Function $x \to 1$) 0

If $\Gamma \vdash e : \tau$ and x is not free in e then $\Gamma \setminus x \vdash e : \tau$.

By induction on height of e, then by case analysis on form of e.

$P(0)$: e is an integer or a booleam or a variable.

If e is an integer, then proof is of form

$\Gamma \vdash e : \mathrm{Int}$ b/c Int rule applies. Int does not use Γ.

So $\Gamma \setminus x \vdash e : \mathrm{Int}$ by Int rule.

If e is booleam, proceeds as above.

If e is variable, then e is x' s.t. $x \neq x'$. Hyp rule must apply:

proof is $\dfrac{x' : \tau \in \Gamma}{\Gamma \vdash x' : \tau}$. $x' : \tau \in (\Gamma \setminus x)$. So, by Hyp rule,

$$\dfrac{x' : \tau \in (\Gamma \setminus x)}{\Gamma \setminus x \vdash x' : \tau}.$$

$P(0) \ldots P(n)$ implies $P(n+1)$

e is of form $e_1 + e_2$, ...

Suppose $e = \mathrm{Let}\ x' : \tau' = e_1\ \mathrm{In}\ e_2$. Then Let Rule applies.

Proof: $$\dfrac{\Gamma \vdash e_1 : \tau' \qquad \Gamma, x' : \tau' \vdash e_2 : \tau}{\Gamma \vdash \mathrm{Let}\ x' : \tau' = e_1\ \mathrm{In}\ e_2 : \tau}$$

By ind, $\Gamma \setminus x' : e_1 : \tau'$.

Either $x = x'$ or not. If $x = x'$, then $(\Gamma \setminus x'), x' : \tau' \vdash e_2 : \tau$

$\Gamma, x' : \tau' \vdash e_2 : \tau$.