TFb Soundness: If $e \Rightarrow v$ and $\Gamma \vdash e:\tau$ then $\Gamma \vdash v:\tau$.

Proof: by induction on the height of the proof of $\Gamma\, \cancel{\vdash e:\tau}$ $e \Rightarrow v$ and then by case analysis on the proof rule used.

Case of Int Rule: $e = n \in \mathbb{Z}$, $\tau = \text{Int}$. So the proof $e \Rightarrow v$ must use the Value Rule, and $e = n = v$. So by the Int Rule, $\Gamma \vdash v : \text{Int}$.

Case of Bool Rule: $e = \text{True}$ or $e = \text{False}$. Also $\tau = \text{Bool}$. So the proof of $e \Rightarrow v$ must use the Value Rule and so $e = v$. By the Bool Rule, $\Gamma \vdash v : \text{Bool}$.

Case of Function Rule: e is of form Function $x \to e'$. τ has form $\tau_1 \to \tau_2$. The proof of $e \Rightarrow v$ must use the Value Rule, so $e = v$. So $\Gamma \vdash v : \tau$.

Case of Plus Rule: e has form $e_1 + e_2$ and $\tau = \text{Int}$. By premises, we know $\Gamma \vdash e_1 : \text{Int}$ and $\Gamma \vdash e_2 : \text{Int}$. Because e is an addition, $e \Rightarrow v$ must use the Plus Rule, so from its premises, we know $e_1 \Rightarrow v_1$ and $e_2 \Rightarrow v_2$. By inductive hypothesis, we know $\Gamma \vdash v_1 : \text{Int}$ and $\Gamma \vdash v_2 : \text{Int}$. We also know $v_1 \in \mathbb{Z}$ and $v_2 \in \mathbb{Z}$ and v is the sum of v_1 and v_2, so $v \in \mathbb{Z}$. By the Int Rule, $\Gamma \vdash v : \text{Int}$.

Case of If Rule: e has form If e_1 Then e_2 Else e_3. By premises, we know $\Gamma \vdash e_1 : \text{Bool}$, $\Gamma \vdash e_2 : \tau$, and $\Gamma \vdash e_3 : \tau$. Because of the form of e, the proof $e \Rightarrow v$ must use either the If-Then or If-Else rule.

Subcase of If-Then: $e_1 \Rightarrow \text{True}$ and $e_2 \Rightarrow v$. By the inductive hypothesis, $\Gamma \vdash v : \tau$.

Subcase of If-Else: $e_1 \Rightarrow \text{False}$ and $e_3 \Rightarrow v$. By the inductive hypothesis, $\Gamma \vdash v : \tau$.

Case of Application:

We know e has form $(e_1\ e_2)$. We also know that $\Gamma \vdash e_1 : \tau_1 \to \tau_2$ and $\Gamma \vdash e_2 : \tau_1$ and $\tau_2 = \tau$. Because $e = e_1\ e_2$, the Application Rule must have been used in the proof of $e \Rightarrow v$. By premises, we know $e_1 \Rightarrow \text{Function } x{:}\tau_1 \to e'$, $e_2 \Rightarrow v_2$, and $e'[v_2/x] \Rightarrow v$.

By ind. hyp., we know $\Gamma \vdash (\text{Function } x{:}\tau_1 \to e') : \tau_1 \to \tau_2$.

The Function Rule must be used to prove this, so we know $\Gamma, x{:}\tau_1 \vdash e' : \tau_2$. By ind. hyp. $\Gamma \vdash v_2 : \tau_1$.

By Substitution Lemma, $\Gamma \vdash e'[v_2/x] : \tau_2$.

By ind. hyp., $\Gamma \vdash v : \tau_2$.

$$\frac{\Gamma \vdash e_1 : \tau_1 \to \tau_2 \qquad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1\ e_2 : \tau_2}$$

$$\frac{e_1 \Rightarrow \text{Function } x \to e' \qquad e_2 \Rightarrow v_2 \qquad e'[v_2/x] \Rightarrow v}{e_1\ e_2 \Rightarrow v}$$

$$\frac{\Gamma, x{:}\tau \vdash e : \tau'}{\Gamma \vdash (\text{Function } x{:}\tau \to e) : \tau \to \tau'}$$

Fb

$\Longrightarrow$
- conv
- div
  - stuck
  - inf. loop

Fb☹

$\Longrightarrow$
- conv
  - value
  - stuck
- div
  - inf. loop

$v ::= \cdots \mid ☹$

$$\frac{e_1 \Rightarrow ☹}{e_1 + e_2 \Longrightarrow ☹}$$

$$\frac{e_1 \Rightarrow v \quad e_2 \Rightarrow ☹}{e_1 + e_2 \Rightarrow ☹}$$