"BOOL is normalizing."

$$\forall e.\ \exists v.\ \underbrace{e \Rightarrow v}_{\text{defn is inductive}}$$

"Strong induction"

* $P(0)$
* $P(0) \wedge \cdots \wedge P(n)$ implies $P(n+1)$

By induction on the height of e.

$P(n) \equiv \forall e.\ \exists v.\ e \Rightarrow v$ when e has height n

BOOL is deterministic.

$\forall e.$ If $e \Rightarrow v_1$ and $e \Rightarrow v_2$ then $v_1 = v_2$.

Proof by induction on the height of e.

Base case: e has height 0 so either e = True or e = False. We know $e \Rightarrow v_1$. If e = True or e = False, the only rule which can prove $e \Rightarrow v_1$ is the Value rule, so $e = v_1$. Likewise, $e \Rightarrow v_2$ so $e = v_2$. Then $v_1 = v_2$. ✓

Inductive step: e has height > 0. So e has form e_1 And e_2 or e_1 Or e_2 or Not e_1. By case analysis:

- Suppose e = Not e_1. Then Not $e_1 \Rightarrow v_1$ and Not $e_1 \Rightarrow v_2$. The proof of Not $e_1 \Rightarrow v_1$ must use the Not rule. So we know $e_1 \Rightarrow v_1'$ where v_1 is the logical negation of v_1'. Likewise, because Not $e_1 \Rightarrow v_2$, $e_1 \Rightarrow v_2'$ where v_2 is the logical negation of v_2'. By the inductive hypothesis: because $e_1 \Rightarrow v_1'$ and $e_1 \Rightarrow v_2'$ then $v_1' = v_2'$. Since v_1' is the logical negation of v_1 and $v_2' = v_1'$ is the logical negation of v_2. So $v_1 = v_2$.
- And ....
- Or ....

TFb Strengthening Lemma

If $\Gamma, x:\tau' \vdash e : \tau$ and x is not free in e then $\Gamma \vdash e : \tau$.

$$\dfrac{\dfrac{}{\emptyset \vdash 1 : Int} \qquad \dfrac{\dfrac{}{\{a:Int\} \vdash 2 : Int} \qquad \dfrac{}{\{a:Int\} \vdash 3 : Int}}{\{a:Int\} \vdash 2+3 : Int}}{\emptyset \vdash Let\ a:Int = 1\ \ In\ \ 2+3 : Int}$$

$$\dfrac{\dfrac{}{\emptyset \vdash 2 : Int} \qquad \dfrac{}{\emptyset \vdash 3 : Int}}{\emptyset \vdash 2+3 : Int} \qquad\qquad \dfrac{\dfrac{}{\{a:Int\} \vdash 2 : Int} \qquad \dfrac{}{\{a:Int\} \vdash 3 : Int}}{\{a:Int\} \vdash 2+3 : Int}$$

Prove the TFb Strengthening Lemma by induction on the height of e.

Proof by induction on the height of e.

Base case: height of $e = 0$. So e is of the form True, False, $\mathbb{Z}$, or x'.

By case analysis:

- If $e =$ True then the proof $\Gamma, x:\tau' \vdash e : \tau$ must use the Bool rule. Then $\tau =$ Bool. By the Bool rule, $\Gamma \vdash$ True : Bool for any Γ, so this case is finished.
- If $e =$ False, then the same argument holds.
- If $e \in \mathbb{Z}$, then the proof $\Gamma, x:\tau' \vdash e : \tau$ must use the Int rule. Then $\tau =$ Int. By the Int rule, $\Gamma \vdash e :$ Int for any Γ. ✓
- If $e = x'$ then $x' \neq x$ because x is not free in e. The proof of $\Gamma, x':\tau' \vdash e : \tau$ must use the Hypothesis rule. The premise is that $(x' : \tau) \in (\Gamma, x : \tau')$ Since $x \neq x'$, $(x' : \tau) \in \Gamma$. So, by Hyp rule, $\Gamma \vdash e : \tau$.

Inductive case: height of $e > 0$. By case analysis:

- If $e =$ Not e' then the proof $\Gamma, x:\tau' \vdash e : \tau$ uses the Not rule. So $\tau =$ Bool and $\Gamma, x:\tau' \vdash e' :$ Bool. So by the inductive hypothesis, $\Gamma \vdash e' :$ Bool. By the Not rule, $\Gamma \vdash$ Not e' : Bool and so this case is finished.