1. Circle

Consider the following grammar for a language called Circle:

$$\begin{array}{rcl} e & ::= & \texttt{Duck}\ e \mid v \\ v & ::= & \texttt{Goose} \end{array}$$

We give an operational semantics for this language:

$$\textsc{Duck}\ \frac{e \Rightarrow v}{\texttt{Duck}\ e \Rightarrow v} \qquad \textsc{Goose}\ \frac{}{\texttt{Goose} \Rightarrow \texttt{Goose}}$$

We state the following theorem:

Theorem 1. *Circle is normalizing; that is,* $\forall e.\, \exists v.\, e \Rightarrow v$.

Prove Theorem 1.

2. Robot

Consider the following grammar for a language called Robot:

$$
\begin{array}{rcl}
e & ::= & \texttt{Beep}\ e\ e \mid v \\
v & ::= & \texttt{Boop}
\end{array}
$$

We give an operational semantics for this language:

$$
\textsc{Beep}\ \frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2}{\texttt{Beep}\ e_1\ e_2 \Rightarrow v_2} \qquad\qquad \textsc{Boop}\ \frac{}{\texttt{Boop} \Rightarrow \texttt{Boop}}
$$

We state the following theorem:

Theorem 2. *Robot is normalizing; that is,* $\forall e. \exists v. e \Rightarrow v$.

Prove Theorem 2.

3. CoinFlip

Consider the following grammar for a language called CoinFlip:

$$\begin{array}{rcl} e & ::= & \texttt{Flip}\ e \mid \texttt{Stop} \\ v & ::= & \texttt{Heads} \mid \texttt{Tails} \end{array}$$

We define the *opposite* of values v in CoinFlip as follows: the opposite of `Heads` is `Tails` and the opposite of `Tails` is `Heads`.

We give the following operational semantics for CoinFlip, which we denote $e \overset{\text{H}}{\Rightarrow} v$:

$$\textsc{Stop}\ \frac{}{\texttt{Stop} \overset{\text{H}}{\Rightarrow} \texttt{Heads}} \qquad \textsc{Flip}\ \frac{e \overset{\text{H}}{\Rightarrow} v \qquad v' \text{ is the opposite of } v}{\texttt{Flip}\ e \overset{\text{H}}{\Rightarrow} v'}$$

We also give *another* operational semantics for CoinFlip, this time denoted $e \overset{\text{T}}{\Rightarrow} v$:

$$\textsc{Stop}\ \frac{}{\texttt{Stop} \overset{\text{T}}{\Rightarrow} \texttt{Tails}} \qquad \textsc{Flip}\ \frac{e \overset{\text{T}}{\Rightarrow} v \qquad v' \text{ is the opposite of } v}{\texttt{Flip}\ e \overset{\text{T}}{\Rightarrow} v'}$$

Note that these relations give two different ways of evaluating expressions. We state the following theorem:

Theorem 3. *If* $e \overset{\text{H}}{\Rightarrow} v$ *then* $\texttt{Flip}\ e \overset{\text{T}}{\Rightarrow} v$*.*

Prove Theorem 3.

4. Addsolute

Consider the following grammar for a language called Addsolute:

$$\begin{array}{rcl} e & ::= & v \mid e \texttt{ + } e \mid \texttt{-}e \mid \texttt{Abs } e \\ v & ::= & \texttt{0} \mid \texttt{1} \mid \texttt{2} \mid \ldots \end{array}$$

We give this language the following operational semantics:

$$\textsc{Value}\ \frac{}{v \Rightarrow v} \qquad \textsc{Negate}\ \frac{e \Rightarrow v \qquad v' \text{ is the arithmetic negation of } v}{\texttt{-}e \Rightarrow v'}$$

$$\textsc{Plus}\ \frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2 \qquad v \text{ is the arithmetic sum of } v_1 \text{ and } v_2}{e_1 \texttt{ + } e_2 \Rightarrow v}$$

$$\textsc{Abs Pos}\ \frac{e \Rightarrow v \qquad v \geq 0}{\texttt{Abs } e \Rightarrow v} \qquad \textsc{Abs Neg}\ \frac{e \Rightarrow v \qquad v < 0 \qquad \texttt{-}e \Rightarrow v'}{\texttt{Abs } e \Rightarrow v'}$$

We state the following theorem:

Theorem 4. *Addsolute is deterministic; that is, if* $e \Rightarrow v$ *and* $e \Rightarrow v'$ *then* $v = v'$.