$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

$n+1$ $n+1$ $\cdots$ $n+1$

$\frac{n}{2} : n+1$

Mathematical Induction

$P(n)$ — inductive hypothesis

$$\boxed{P(n) \equiv \sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$$

Base case: Prove $P(k)$ (usually for small k)

Inductive case: Prove that, if $P(n)$ is true, then $P(n+1)$ is true

Ind Hyp. $P(n) \equiv \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Base case: $P(1) \equiv \sum_{i=1}^{1} i = \frac{1(1+1)}{2}$

$$1 = \frac{1 \cdot 2}{2} = \frac{2}{2} = 1$$

Inductive case: $P(n) \Rightarrow P(n+1)$

$$P(n) \equiv \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$P(n+1) \equiv \sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

$$\left(\sum_{i=1}^{n} i\right) + n + 1 = \frac{n(n+1)}{2} + (n+1)$$

$$(1 + 2 + \cdots + n) + n + 1 = \frac{n(n+1)}{2} + n + 1$$

$$\sum_{i=1}^{n+1} i = \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1) + 2n + 2}{2} = \frac{n^2 + n + 2n + 2}{2}$$

$$= \frac{n^2 + 3n + 2}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

$$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow P(4) \Rightarrow \cdots$$

$$P(n) \equiv \sum_{i=0}^{n-1} 2^i = 2^n - 1$$

Base case: $P(1) \equiv \sum_{i=0}^{1-1} 2^i = 2^1 - 1$

$$2^0 = 2^1 - 1$$
$$1 = 2 - 1 \quad \checkmark$$

Inductive case: $P(n) \Rightarrow P(n+1)$

$$P(n) \equiv \sum_{i=0}^{n-1} 2^i = 2^n - 1 \qquad\qquad P(n+1) \equiv \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

$$\sum_{i=0}^{n} 2^i = 2^n - 1 + 2^n$$
$$= 2 \cdot 2^n - 1$$
$$= 2^{n+1} - 1 \quad \checkmark$$

Prove for all positive n:

Suppose some $a > 1$.

$$\boxed{\text{Then } \sum_{i=1}^{n} a^i = \frac{a(a^n - 1)}{a - 1}}$$

$$P(n) \equiv \sum_{i=1}^{n} a^i = \frac{a(a^n - 1)}{a - 1}$$

Base Case: $P(1) \equiv \sum_{i=1}^{1} a^i = \frac{a(a^1 - 1)}{a - 1}$

$$a^1 = \frac{a\cancel{(a-1)}}{\cancel{a-1}} \quad \checkmark$$

Inductive Case: $P(n) \Rightarrow P(n+1)$

$$P(n) \equiv \sum_{i=1}^{n} a^i = \frac{a(a^n - 1)}{a - 1} \qquad\qquad P(n+1) \equiv \sum_{i=1}^{n+1} a^i = \frac{a(a^{n+1} - 1)}{a - 1}$$

$$\sum_{i=1}^{n+1} a^i = \frac{a(a^n - 1)}{a - 1} + a^{n+1}$$

$$= \frac{a(a^n - 1)}{a - 1} + \frac{a^{n+1} \cdot (a-1)}{a - 1} = \frac{a(a^n - 1) + a^{n+1} \cdot (a-1)}{a - 1}$$

$$= \frac{a(a^n - 1) + a^{n+2} - a^{n+1}}{a - 1} = \frac{a(a^n - 1) + a(a^{n+1} - a^n)}{a - 1} = \frac{a(\cancel{a^n} - 1 + a^{n+1} - \cancel{a^n})}{a - 1}$$

```
Function summate(x):
    If x = 1 Then
        Return 1
    Else
        Return x + summate(x-1)
    End If
EndFunction
```

$P(n) \equiv$ summate(n) returns the value $\sum_{i=1}^{n} i$.

Base Case: $P(1) \equiv$ summate(1) returns the value 1.

summate(1) sets $x = 1$. Next, we consider the If condition $1 = 1$ and enter the Then block. This block returns 1 immediately.

Inductive Case: $P(n) \Rightarrow P(n+1) \equiv$ If summate(n) returns the value $\sum_{i=1}^{n} i$, then summate$(n+1)$ returns the value $\sum_{i=1}^{n+1} i$.

summate$(n+1)$ sets x to $n+1$. Next, we consider the If condition. Because n is positive and $x = n+1$, $x \neq 1$. So we enter the Else block. We assume by induction that summate(n) returns $\sum_{i=1}^{n} i$. Because $x = n+1$, $x - 1 = n$. So summate$(x-1)$ returns $\sum_{i=1}^{n} i$. We return summate$(x-1) + x$. This is $\left(\sum_{i=1}^{n} i\right) + x = \left(\sum_{i=1}^{n} i\right) + n + 1 = \sum_{i=1}^{n+1} i$. ✓

$P(n) \equiv$ My object invariants hold after n method calls.

Base Case: $P(0) \equiv$ After construction, my object .....

Inductive Case: $P(n) \Rightarrow P(n+1) \equiv$

If, after n calls, my obj. ....,
then after $n+1$ calls, my obj. .....