CS46, Swarthmore College, Spring 2018
Lab 9 (due Wednesday 2 May)
Name: YOUR NAME(S) HERE

1. Suppose $A \leq_p B$ for all $A, B \in \mathrm{NP}$. This implies amongst other things that if $A \leq_p B$ then $B \leq_p A$ for any $A, B \in \mathrm{NP}$. Show that if the first statement is true then $\mathrm{P} = \mathrm{NP}$.

2. (Sipser 7.18) Show that if $\mathrm{P} = \mathrm{NP}$, then every language $A \in \mathrm{P}$ is NP-complete except $A = \emptyset$ and $A = \Sigma^*$.

3. Give a polynomial time reduction from $3 - \textsc{Color}$ to $3 - \textsc{Sat}$. You may want to use the variables V_{ij} to indicate that vertex V_i has color $j, 1 \leq j \leq 3$. If we know $3 - \textsc{Sat}$ is NP-complete and $3 - \textsc{Color}$ is in NP, what does this reduction tell us?

4. (Lewis & Papadimitriou 6.3.3)

$$2 - \textsc{Sat} = \left\{ \langle \varphi \rangle \;\middle|\; \begin{array}{l} \varphi \text{ is a satisfiable formula in conjunctive normal form} \\ \text{with exactly two literals per clause} \end{array} \right\}$$

   We will prove that $2 - \textsc{Sat} \in \mathrm{P}$.

   Any clause $(x \vee y)$ with two literals can be thought of as two implications $\overline{x} \Rightarrow y$ and $\overline{y} \Rightarrow x$. The clause $(x \vee x)$ can be thought of as $\overline{x} \Rightarrow x$. If we then consider $x \Rightarrow y$ as a directed edge from vertex x to vertex y, we can construct an “implication graph” from any 2-CNF formula φ.

   (a) Show that a 2-CNF formula is unsatisfiable if and only if there is a variable x such that in the implication graph, there is a path from x to $\overline{x}$ and from $\overline{x}$ to x.
   (b) Design an algorithm based on this fact to show that $2 - \textsc{Sat} \in \mathrm{P}$.

5. Recall that a **vertex cover** in a graph G is a subset of vertices where every edge of G has at least one endpoint in the subset.

$$\textsc{VertexCover} = \{\langle G, k\rangle \mid G \text{ has a } k\text{-node vertex cover}\}$$

   Theorem 7.44 says that $\textsc{VertexCover}$ is NP-complete.

   An **independent set** in a graph G is a subset of vertices with no edges between them.

$$\textsc{IndependentSet} = \{\langle G, k\rangle \mid G \text{ contains an independent set of } k \text{ vertices}\}$$

   We will show that $\textsc{IndependentSet}$ is NP-complete.

   (a) Prove that $\textsc{IndependentSet} \in \mathrm{NP}$.
   (b) Prove that $\textsc{IndependentSet}$ is NP-hard. (Hint: reduce from $\textsc{VertexCover}$, though you can use any)